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Arbitrary spin field equations and anomalies in the Riemann–Cartan space–time

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Abstract. We obtain the new generalised wave equations for the fields of arbitrary spin in the Riemann–Cartan space–time. The arising consistency conditions are shown to be more restrictive than in the Riemannian space, eliminating the minimal coupling of torsion to the higher-spin fields.

1. Introduction

The Riemann–Cartan geometry arises on a space–time within the framework of the Poincaré gauge theory of gravity (Sciama (1962), Kibble (1961); for a review of earlier works see Hehl *et al* (1976), Ivanenko (1980); for the modern fibre bundle approach see Luehr and Rosenbaum (1980), Tseytlin (1982); the last reference contains a good bibliography of recent works). The gauge gravitational field is described by the tetrad (vierbein) fields h^a_μ and the local Lorentz connection $\tilde{\Gamma}^{ab}{}_\mu = -\tilde{\Gamma}^{ba}{}_\mu$ (our conventions are summarised in the appendix). These define the world affine connection $\tilde{\Gamma}^\alpha_{\beta\mu} = h^\alpha_a h^b_\beta \tilde{\Gamma}^a_{b\mu} + h^\alpha_a \partial_\mu h^a_\beta$, which is compatible with the metric ($\tilde{\nabla}_\mu g_{\alpha\beta} = 0$) but possesses non-zero torsion $Q^{\lambda}_{\cdot\mu\nu} = \frac{1}{2}(\tilde{\Gamma}^\lambda_{\mu\nu} - \tilde{\Gamma}^\lambda_{\nu\mu}) = \tilde{\Gamma}^\lambda_{[\mu\nu]}$. Such a geometrical structure is called the Riemann–Cartan space–time U_4.

Recently several authors (Goldthorpe 1980, Kimura 1981a, b, Nieh and Yan 1982, Obukhov 1982, 1983) have achieved some progress in understanding the so-called spectral geometry of the Riemann–Cartan space–time, which relates zero modes of differential operators for the quantised fields with the geometrical and topological properties of the underlying manifold. The next step is to use these results and to generalise the index theorems (see Christensen and Duff 1979) to the case of non-zero torsion. This can shed new light on the possible role of non-trivial microscopic topology of space–time, e.g. of the gravitational instantons with torsion (Tseytlin 1982), in quantum physics.

In this connection it is necessary to consider the theory of arbitrary spin fields in U_4. Here we investigate the natural minimal coupling of matter to the gauge gravitational fields. We understand the minimal coupling principle in the geometrical sense (Benn *et al* 1980): dynamics of an arbitrary spin field, which can be considered as a (local) section of the corresponding (here, spinor) bundle, is determined by the bundle connection. In practice this means that the flat space theory should be rewritten in covariant form (in terms of covariant derivatives instead of the partial ones etc). One should not, however, confuse this rule with the case of the gauge fields, which are connections on the bundle and thus are not on an equal footing with the other

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matter fields. This latter case was already investigated earlier (Benn *et al* 1980) and it was shown that in a curved non-Riemannian space–time the gauge fields do not couple directly to the SO(4) Lorentz connection $\tilde{\Gamma}^{a}_{b\mu}$. In the present paper we consider the general case of spinor fields with arbitrary mass and spin in the Riemann–Cartan background space–time. We derive the new generalised field equations and the consistency conditions, correcting several shortcomings of previous works of Kimura (1981a, b), Goldthorpe (1980) and Barth and Christensen (1983).

2. Arbitrary spin equations

The two-component spinor formalism is the most convenient mathematical framework for the description of the arbitrary spin fields. Many useful details about the spinors in flat and curved space–times can be found for instance in Christensen and Duff (1979), Barth and Christensen (1983) and references therein. So we just summarise our conventions. Since in perspective we are interested in the gravitational instanton effects, we consider the Euclidean sector for the metric, i.e. locally $g_{\mu\nu} = \mathrm{diag}(+1, +1, +1, +1)$. The flat space generalised Pauli matrices $\sigma^{a}{}_{AB'} = (I, -\mathrm{i}\boldsymbol{\sigma})$ satisfy the canonical decomposition

$$\sigma^{a}{}_{AC'}\sigma^{b}{}_{B\cdot}{}^{C'} = \delta^{ab}\varepsilon_{AB} + \overset{(+)}{S}{}^{ab}{}_{AB}, \qquad \sigma^{a}{}_{CA'}\sigma^{bC}{}_{\cdot B'} = \delta^{ab}\varepsilon_{A'B'} + \overset{(-)}{S}{}^{ab}{}_{A'B'}, \tag{1}$$

where

$$\overset{(+)}{S}{}^{ab}{}_{AB} = \sigma^{[a}{}_{AC'}\sigma^{b]}{}_{B\cdot}{}^{C'}, \qquad \overset{(-)}{S}{}^{ab}{}_{A'B'} = \sigma^{[a}{}_{CA'}\sigma^{b]C}{}_{\cdot B'}.$$

Spinor indices are raised and lowered with the help of $\varepsilon_{AB} = \varepsilon^{AB} = \left(\begin{smallmatrix} 0 & 1 \\ -1 & 0 \end{smallmatrix}\right)$, e.g. $\phi^{A} = \varepsilon^{AB}\phi_{B}$, $\phi_{A} = \phi^{B}\varepsilon_{BA}$.

From the definition (1) one can see the symmetry and duality properties of the fundamental spin-tensors

$$\begin{aligned} &\overset{(+)}{S}{}^{ab}{}_{AB} = \overset{(+)}{S}{}^{ab}{}_{BA}, \qquad \overset{(-)}{S}{}^{ab}{}_{A'B'} = \overset{(-)}{S}{}^{ab}{}_{B'A'}, \\ &\overset{(+)}{S}{}^{ab}{}_{AB} = \tfrac{1}{2}\varepsilon^{ab}{}_{\cdot\cdot cd}\overset{(+)}{S}{}^{cd}{}_{AB}, \qquad \overset{(-)}{S}{}^{ab}{}_{A'B'} = -\tfrac{1}{2}\varepsilon^{ab}{}_{\cdot\cdot cd}\overset{(-)}{S}{}^{cd}{}_{A'B'}, \end{aligned} \tag{2}$$

where ε^{abcd} is the totally antisymmetric Levi-Civita tensor, $\varepsilon^{0123} = +1$. Space–time dependent matrices are $\sigma^{\mu}{}_{AB'} = h^{\mu}_{a}\sigma^{a}{}_{AB'}$.

As is well known (Umezawa 1956), in flat space particles with spin S are described by one of equivalent theories (there are S theories for bosons and $S+\frac{1}{2}$ for fermions) for the spinors, which transform according to an irreducible (A, B) representation of the Lorentz group $(A + B = S)$.

At first, for definiteness, let us consider a particular case: the theory of fermions, $S = k + \frac{1}{2}$, $k = 0, 1, \ldots$, determined by the Lagrangian

$$\begin{aligned} L = \tfrac{1}{2}\mathrm{i}(&\bar{\varphi}^{A'_0A'_1\ldots A'_kB_1\ldots B_k}\overleftrightarrow{\partial}_{B_0A'_0}\varphi^{B_0}{}_{\cdot B_1\ldots B_kA'_1\ldots A'_k} + \bar{\chi}^{A'_1\ldots A'_kB_0B_1\ldots B_k}\overleftrightarrow{\partial}_{B_0A'_0}\chi_{B_1\ldots B_k\cdot}{}^{A'_0}{}_{A'_1\ldots A'_k}) \\ &- m(\bar{\varphi}^{A'_0A'_1\ldots A'_kB_1\ldots B_k}\chi_{B_1\ldots B_kA'_0\ldots A'_k} + \bar{\chi}^{A'\ldots A'_kB_0B_1\ldots B_k}\varphi_{B_0B_1\ldots B_kA'_1\ldots A'_k}). \end{aligned} \tag{3}$$

Here we denote $\partial_{AB'} = \sigma^{a}{}_{AB'}\partial_{a}$, $\bar{\varphi}\overleftrightarrow{\partial}\varphi = \bar{\varphi}(\partial\varphi) - (\partial\bar{\varphi})\varphi$, and $\bar{\varphi}$ is the complex conjugate spinor.

We want to stress here that contrary to earlier attempts we start from the action principle. This is a crucial point since introduction of the gravitational interaction (i.e. covariantisation of the theory) directly in the flat space field equations is to a great extent an ambiguous procedure (for a related discussion see Aragone and Deser (1980)), and in particular it can lead to non-self-adjoint operators. Indeed, this is the case for the field equations proposed by Goldthorpe (1980, equations (2.17) and appendix B), by Kimura (1981b, equations (2.13)–(2.15)) and quite recently by Barth and Christensen (1983, equations (6.1)–(6.44)). For example, when the scalar field equation $-\partial_\mu\partial^\mu\varphi=0$ is generalised in U_4 according to their prescription, it reads $-\tilde{\nabla}_\mu\tilde{\nabla}^\mu\varphi=0$ (Barth and Christensen 1983, equation (6.1)). But the operator $-\tilde{\square}=-\tilde{\nabla}_\mu\tilde{\nabla}^\mu=-\nabla_\mu\nabla^\mu-2Q^\lambda{}_{.\mu\lambda}\nabla^\mu$ is obviously not self-adjoint with respect to the usual scalar product $(\varphi,\psi)=\int \mathrm{d}^4x(\det g_{\mu\nu})^{1/2}\bar{\varphi}\psi$. The same is true for the higher spins. The non-trivial point, overlooked by the above authors, is that the Gauss theorem in U_4 reads $\int_V \mathrm{d}^4x(\det g_{\mu\nu})^{1/2}(\tilde{\nabla}_\mu-2Q^\lambda{}_{.\mu\lambda})A^\mu=\int_{\partial V}\mathrm{d}\sigma_\mu A^\mu$ (see Hehl *et al* 1976).

In our approach we have no risk of obtaining unphysical results, as starting from the action principle one automatically gets self-adjoint operators.

Now we return to the theory under consideration and replace, according to the minimal coupling principle, the flat quantities in (3) by the Riemann–Cartan covariant ones: $\sigma^a{}_{AB'}\to\sigma^\mu{}_{AB'}=h^\mu_a\sigma^a{}_{AB'}$; $\partial_\mu\to\tilde{\nabla}_\mu$. The full Riemann–Cartan derivative is $\tilde{\nabla}_\mu=\partial_\mu+\omega_\mu+I^\alpha_\beta\tilde{\Gamma}^\beta_{\alpha\mu}$, where I^α_β are the generators of the general coordinate transformations and $\omega_\mu=\frac{1}{2}\Sigma_{ab}\tilde{\Gamma}^{ab}{}_\mu$ is the spinor connection with Σ_{ab} as SO(4) generators. The resulting action describes fermions on the U_4 background and yields after variation by the spinors $\varphi(\frac{1}{2}(k+1),\frac{1}{2}k)$ and $\chi(\frac{1}{2}k,\frac{1}{2}(k+1))$ the following wave equations:

$$
\begin{aligned}
&\mathrm{i}\overset{*}{\nabla}_{A_0.}{}^{B_0'}\varphi^{A_0A_1\ldots A_kB_1'\ldots B_k'}=m\chi^{A_1\ldots A_kB_0'\ldots B_k'},\\
&\mathrm{i}\overset{*}{\nabla}{}^{A_0}{}_{.B_0'}\chi^{A_1\ldots A_kB_0'B_1'\ldots B_k}=m\varphi^{A_0A_1\ldots A_kB_1'\ldots B_k'}
\end{aligned}\tag{4}
$$

where we denoted the operator $\overset{*}{\nabla}_{AB'}=\sigma^\mu{}_{AB'}\overset{*}{\nabla}_\mu$, $\overset{*}{\nabla}_\mu=\tilde{\nabla}_\mu-Q_\mu$, $Q_\mu=Q^\lambda{}_{.\mu\lambda}$. The trace of the torsion appeared due to the use of the mentioned theorem in U_4.

Applying $\overset{*}{\nabla}$ to (4) one gets the second-order equations

$$
\overset{*}{\nabla}{}^{A_0}_{.C'}\overset{*}{\nabla}_{D.}{}^{C'}\varphi^{DA_1\ldots A_kB_1'\ldots B_k'}+m^2\varphi^{A_0A_1\ldots A_kB_1'\ldots B_k'}=0,\tag{5}
$$

$$
\overset{*}{\nabla}_{C.}{}^{B_0'}\overset{*}{\nabla}{}^{C}_{.D'}\chi^{A_1\ldots A_kD'B_1'\ldots B_k'}+m^2\chi^{A_1\ldots A_kB_0'B_1'\ldots B_k'}=0,\tag{6}
$$

which show that the U_4 gravitational coupling is in general inconsistent. Indeed, since the spinors φ and χ are totally symmetric in both types of indices, contraction of (5) with $\varepsilon_{A_0A_1}$ and of (6) with $\varepsilon_{B_0'B_1'}$ gives additional constraints

$$
\overset{(+)}{S}{}^{\mu\nu}{}_{A_0A_1}[\overset{*}{\nabla}_\mu\overset{*}{\nabla}_\nu]\varphi^{A_0A_1A_2\ldots A_kB_1'\ldots B_k'}=0,\tag{7}
$$

$$
\overset{(-)}{S}{}^{\mu\nu}{}_{B_0'B_1'}[\overset{*}{\nabla}_\mu\overset{*}{\nabla}_\nu]\chi^{A_1\ldots A_kB_0'B_1'\ldots B_k'}=0.\tag{8}
$$

These consistency conditions (like the analogous ones in the Riemannian case) provide some constraints on the background geometry and yield inequivalence of different higher-spin theories (which were equivalent in flat space). Supposing that (7)–(8) are satisfied, one can symmetrise the first terms in (5)–(6).

Up to this point we have considered the particular theory for $S=k+\frac{1}{2}$, described in terms of $(\frac{1}{2}(k+1),\frac{1}{2}k)$ and $(\frac{1}{2}k,\frac{1}{2}(k+1))$ representations. However, the extension to the general case is now obvious and straightforward: repeating the above procedure for an arbitrary (A,B) representation, we finally obtain the following generalisation of the Riemannian result (Christensen and Duff 1979).

For the fermion field $\phi(A, B)$, which transforms according to the irreducible (A, B) representation of SO(4), the new non-Riemannian self-adjoint second-order operator Δ acting on $\phi(A, B)$ is

$$
\begin{aligned}
(\Delta\phi)^{A_1\ldots A_{2A}B'_1\ldots B'_{2B}} &= -(\overset{*}{\nabla}_\mu \overset{*}{\nabla}^\mu - m^2)\phi^{A_1\ldots A_{2A}B'_1\ldots B'_{2B}} \\
&\quad + \frac{1}{2A}\sum_{i=1}^{2A} \overset{(+)}{S}{}^{\mu\nu A_i}{}_D \overset{*}{\nabla}_{[\mu}\overset{*}{\nabla}_{\nu]}\phi^{A_1\ldots D\ldots A_{2A}B'_1\ldots B'_{2B}} \qquad (A>B) \\
&= -(\overset{*}{\nabla}_\mu \overset{*}{\nabla}^\mu - m^2)\phi^{A_1\ldots A_{2A}B'_1\ldots B'_{2B}} \\
&\quad + \frac{1}{2B}\sum_{i=1}^{2B} \overset{(-)}{S}{}^{\mu\nu B'_i}{}_{D'} \overset{*}{\nabla}_{[\mu}\overset{*}{\nabla}_{\nu]}\phi^{A_1\ldots A_{2A}B'_1\ldots D'\ldots B'_{2B}} \qquad (A<B).
\end{aligned} \tag{9}
$$

This is already a symmetrised operator, and since it is the square of the first-order fermion operator of the type (4), the generalised consistency conditions

$$
\overset{(+)}{S}{}^{\mu\nu}{}_{A_1A_2}[\overset{*}{\nabla}_\mu \overset{*}{\nabla}_\nu]\phi^{A_1A_2\ldots A_{2A}B'_1\ldots B'_{2B}} = 0, \tag{10a}
$$

$$
\overset{(-)}{S}{}^{\mu\nu}{}_{B'_1B'_2}[\overset{*}{\nabla}_\mu \overset{*}{\nabla}_\nu]\phi^{A_1\ldots A_{2A}B'_1B'_2\ldots B'_{2B}} = 0, \tag{10b}
$$

should be satisfied.

Two comments are in order. First, using the Riemann–Cartan Ricci identity one can see that equations (9) indeed have the supposed (Goldthorpe 1980, Kimura 1981a, b) general structure $\Delta = -\tilde{\nabla}_\mu \tilde{\nabla}^\mu + S^\mu \tilde{\nabla}_\mu + X$, but the matrices S^μ and X differ from the naive generalisation of the Riemannian ones.

Second, when considering bosons ($A+B=$ integer) we can follow along the same lines and obtain the U_4 boson equations in the framework of the general first-order action principle approach. The corresponding second-order operators for the bosons then have the same form (9) as for the fermions, with the proper change of the range of indices. Evidently the consistency conditions (10) are the same for bosons and fermions. Note that in the case of the scalar (0, 0) field we thus obtain the natural result: the torsion and the local Lorentz connection $\tilde{\Gamma}^a_{b\mu}$ do not couple to the spinless scalars.

An alternative way is to continue our equations (4)–(10) to the $A+B=$ integer case with the help of the proper ($\nabla_\mu \to \overset{*}{\nabla}_\mu$) non-Riemannian modification of the Lichnerowicz (1961) operator. This possibility requires a special consideration to which we shall return in a future publication. However, if in this approach we choose the higher-spin second-order boson operator in the form proposed by Christensen and Duff (1979, equation (3.13)), we again get the same (equation (10)) consistency conditions for the bosons.

3. Consistency conditions

Let us now proceed to the detailed analysis of the constraints (10), which generalise the well known Riemannian results (cf Christensen and Duff 1979) to the case of non-zero torsion. Using the U_4 Ricci identity

$$
[\tilde{\nabla}_\mu \tilde{\nabla}_\nu] = I^\beta_\alpha \tilde{R}^\alpha{}_{.\beta\mu\nu} + \tfrac{1}{2}\Sigma_{ab}\tilde{R}^{ab}{}_{..\mu\nu} + 2Q^\alpha{}_{.\mu\nu}\tilde{\nabla}_\alpha,
$$

we can rewrite (10*a*)–(10*b*) as follows;

$$\overset{(+)}{S}{}^{\mu\nu}{}_{A_1A_2}\Big[2(Q^{\alpha}_{.\mu\nu}\tilde{\nabla}_{\alpha}+\tilde{R}_{[\mu\nu]}-\tilde{\nabla}_{[\mu}Q_{\nu]})\phi^{A_1A_2\ldots A_{2A}B'_1\ldots B'_{2B}}$$
$$-\tfrac{1}{4}\tilde{R}_{\alpha\beta\mu\nu}\Big(\sum_{i=3}^{2A}\overset{(+)}{S}{}^{\alpha\beta A_i}{}_{D}\phi^{A_1A_2\ldots D\ldots A_{2A}B'_1\ldots B'_{2B}}$$
$$+\sum_{i=1}^{2B}\overset{(-)}{S}{}^{\alpha\beta B'_i}{}_{D'}\phi^{A_1A_2\ldots A_{2A}B'_1\ldots D'\ldots B'_{2B}}\Big)\Big]=0, \tag{11a}$$

$$\overset{(-)}{S}{}^{\mu\nu}{}_{B'_1B'_2}\Big[2(Q^{\alpha}_{.\mu\nu}\tilde{\nabla}_{\alpha}+\tilde{R}_{[\mu\nu]}-\tilde{\nabla}_{[\mu}Q_{\nu]})\phi^{A_1\ldots A_{2A}B'_1B'_2\ldots B'_{2B}}$$
$$-\tfrac{1}{4}\tilde{R}_{\alpha\beta\mu\nu}\Big(\sum_{i=1}^{2A}\overset{(+)}{S}{}^{\alpha\beta A_i}{}_{D}\phi^{A_1\ldots D\ldots A_{2A}B'_1B'_2\ldots B'_{2B}}$$
$$+\sum_{i=3}^{2B}\overset{(-)}{S}{}^{\alpha\beta B'_i}{}_{D'}\phi^{A_1\ldots A_{2A}B'_1B'_2\ldots D'\ldots B'_{2B}}\Big)\Big]=0. \tag{11b}$$

For consistency of the theory the matter fields ϕ and the gravitational field $(g_{\mu\nu}, \tilde{\Gamma}^{\alpha}_{\beta\mu})$ should satisfy (11). Requiring as usual that these equations should hold for any configuration (values) of the matter fields $\phi(A, B)$, we get the sufficient constraints on the geometrical (i.e. gravitational) quantities in U_4 by putting the curvature and torsion dependent coefficients in (11) equal to zero. As compared with the Riemannian case we see, however, that the higher-spin constraints in U_4 are differential and not algebraic ones.

Now let us obtain the explicit restrictions imposed by (10)–(11) on torsion and curvature. Taking into account the duality properties (2), we get from the first term in (11) that the torsion is (i) unconstrained for $A<1, B<1$, (ii) self-dual (anti-self-dual) if $A<1$, $B\geqslant 1$ $(A\geqslant 1, B<1)$, (iii) zero when $A\geqslant 1, B\geqslant 1$. So contrary to the statement of Kimura (1981b) the higher-spin consistency conditions in U_4 are more restrictive than their Riemannian counterparts, in the sense that for the greater part of matter fields the torsion is completely ruled out.

The other consequences of (10)–(11) are as follows. In view of the identity

$$\tilde{\nabla}_{[\mu}Q_{\nu]}=\tfrac{1}{2}(\tilde{R}_{[\mu\nu]}-\tilde{\nabla}_{\alpha}Q^{\alpha}_{.\mu\nu}+2Q_{\alpha}Q^{\alpha}_{.\mu\nu})$$

(which can be proved directly from the definition of the Ricci tensor in U_4) and assuming that the above torsion duality constraints are already satisfied, the second and third terms in (11*a*) yield $A(A-\frac{1}{2})(\tilde{R}_{[\mu\nu]}+\frac{1}{2}\varepsilon_{\mu\nu..}^{\ \ \ \alpha\beta}\tilde{R}_{[\alpha\beta]})=0$. Respectively from (11*b*) we have

$$B(B-\tfrac{1}{2})(\tilde{R}_{[\mu\nu]}-\tfrac{1}{2}\varepsilon_{\mu\nu..}^{\ \ \ \alpha\beta}\tilde{R}_{[\alpha\beta]})=0.$$

For the further analysis of (10)–(11) we use the irreducible decomposition of the Riemann–Cartan curvature tensor (Gambini and Herrera 1980)

$$\tilde{R}_{\alpha\beta\mu\nu}=\tilde{C}_{\alpha\beta\mu\nu}+\tfrac{1}{2}(g_{\alpha\mu}\tilde{E}_{\beta\nu}-g_{\alpha\nu}\tilde{E}_{\beta\mu}+g_{\beta\nu}\tilde{E}_{\alpha\mu}-g_{\beta\mu}\tilde{E}_{\alpha\nu})+\tfrac{1}{12}(g_{\alpha\mu}g_{\beta\nu}-g_{\alpha\nu}g_{\beta\mu})\tilde{R}$$
$$+\tfrac{1}{12}\varepsilon_{\alpha\beta\mu\nu}D+\tfrac{1}{4}(\varepsilon_{\alpha\beta\mu.}{}^{\lambda}D^{(S)}{}_{\lambda\nu}-\varepsilon_{\alpha\beta\nu.}{}^{\lambda}D^{(S)}{}_{\lambda\mu}-\varepsilon_{\mu\nu\alpha.}{}^{\lambda}D^{(S)}{}_{\lambda\beta}+\varepsilon_{\mu\nu\beta.}{}^{\lambda}D^{(S)}{}_{\lambda\alpha}). \tag{12}$$

Here $\tilde{C}_{\alpha\beta\mu\nu}$ is the 'Weyl tensor' in U_4 with the same symmetry properties (i.e. $(2,0)+(0,2)$ irreducible part of the curvature); $D_{\mu\nu}=\frac{1}{2}\varepsilon^{\alpha\beta\gamma}_{\ \ \ \ ...\mu}\tilde{R}_{\alpha\beta\gamma\nu}$; $D=D_{\mu\nu}g^{\mu\nu}$;

$D^{(S)}{}_{\mu\nu}=D_{(\mu\nu)}-\frac{1}{4}Dg_{\mu\nu}$; $\tilde{E}_{\mu\nu}=\tilde{R}_{\mu\nu}-\frac{1}{4}\tilde{R}g_{\mu\nu}$. Note that for the antisymmetric part of the Ricci tensor we have the identity

$$\tilde{R}_{[\mu\nu]}=\tfrac{1}{2}\varepsilon_{\mu\nu}{}^{\alpha\beta}D_{[\alpha\beta]}. \tag{13}$$

Let us consider the typical term in the first sum in (11*a*),

$$-\tfrac{1}{4}\overset{(+)}{S}{}^{\mu\nu}{}_{AB}\overset{(+)}{S}{}^{\alpha\beta C}{}_{D}\tilde{R}_{\alpha\beta\mu\nu}\phi^{ABD\ldots}.$$

With the help of (12) it reduces to

$$-\tfrac{1}{4}\overset{(+)}{S}{}^{\mu\nu}{}_{AB}\overset{(+)}{S}{}^{\alpha\beta C}{}_{D}\tilde{C}_{\alpha\beta\mu\nu}\phi^{ABD\ldots},$$

provided we use the irreducibility of the spinor ϕ (i.e. its total symmetry in both types of indices) and the well known duality properties (2) and the algebraic identity

$$\overset{(+)}{S}{}^{\alpha\mu AB}\overset{(+)}{S}{}_{\alpha}{}^{\nu}{}_{CD}=2g^{\mu\nu}\delta^{A}{}_{(C}\delta^{B}{}_{D)}-\overset{(+)}{S}{}^{\mu\nu A}{}_{(C}\delta^{B}{}_{D)}-\overset{(+)}{S}{}^{\mu\nu B}{}_{(C}\delta^{A}{}_{D)}$$

(the latter is a direct consequence of (1)).

For the same reasons the typical term in the last sum in (11*a*) can be transformed as follows:

$$-\tfrac{1}{4}\overset{(+)}{S}{}^{\mu\nu}{}_{AB}\overset{(-)}{S}{}^{\alpha\beta C'}{}_{D'}\tilde{R}_{\alpha\beta\mu\nu}\phi^{AB\ldots D'\ldots}=-\tfrac{1}{2}(\tilde{E}^{(S)}_{\mu\nu}-D^{(S)}_{\mu\nu})\overset{(+)}{S}{}^{\alpha\mu}{}_{AB}\overset{(-)}{S}{}_{\alpha}{}^{\nu C'}{}_{D'}\phi^{AB\ldots D'\ldots},$$

where $\tilde{E}^{(S)}_{\mu\nu}=\tilde{E}_{(\mu\nu)}$.

Analogously for (11*b*) we get

$$-\tfrac{1}{4}\overset{(-)}{S}{}^{\mu\nu}{}_{A'B'}\overset{(-)}{S}{}^{\alpha\beta C'}{}_{D'}\tilde{R}_{\alpha\beta\mu\nu}\phi^{\ldots A'B'D'\ldots}=-\tfrac{1}{4}\overset{(-)}{S}{}^{\mu\nu}{}_{A'B'}\overset{(-)}{S}{}^{\alpha\beta C'}{}_{D'}\tilde{C}_{\alpha\beta\mu\nu}\phi^{\ldots A'B'D'\ldots},$$

and

$$-\tfrac{1}{4}\overset{(-)}{S}{}^{\mu\nu}{}_{A'B'}\overset{(+)}{S}{}^{\alpha\beta C}{}_{D}\tilde{R}_{\alpha\beta\mu\nu}\phi^{D\ldots A'B'\ldots}=-\tfrac{1}{2}(\tilde{E}^{(S)}_{\mu\nu}+D^{(S)}_{\mu\nu})\overset{(-)}{S}{}^{\alpha\mu}{}_{A'B'}\overset{(+)}{S}{}_{\alpha}{}^{\nu C}{}_{D}\phi^{D\ldots A'B'\ldots}.$$

Thus taking all intermediate results together, we finally arrive at the following sufficient consistency conditions in U_4:

$$A(A-\tfrac{1}{2})\begin{pmatrix}(A-1)\tilde{C}_{+}\\ Q_{+}\\ D_{+}\\ B(\tilde{E}^{(S)}-D^{(S)})\end{pmatrix}\phi(A,B)=0=B(B-\tfrac{1}{2})\begin{pmatrix}(B-1)\tilde{C}_{-}\\ Q_{-}\\ D_{-}\\ A(\tilde{E}^{(S)}+D^{(S)})\end{pmatrix}\phi(A,B). \tag{14}$$

Here the condensed notation for the self-dual $(+)$ and anti-self-dual $(-)$ parts of antisymmetric tensors is used: $Q_{\pm}{}^{\lambda}{}_{\mu\nu}=\frac{1}{2}(Q^{\lambda}{}_{\mu\nu}\pm\frac{1}{2}\varepsilon_{\mu\nu}{}^{\alpha\beta}Q^{\lambda}{}_{\alpha\beta})$, $D_{\pm\mu\nu}=\frac{1}{2}(D_{[\mu\nu]}\pm\frac{1}{2}\varepsilon_{\mu\nu}{}^{\alpha\beta}D_{[\alpha\beta]})$, $\tilde{C}_{\pm}{}^{\alpha\beta}{}_{\mu\nu}=\frac{1}{2}(\tilde{C}^{\alpha\beta}{}_{\mu\nu}\pm\frac{1}{2}\varepsilon_{\mu\nu}{}^{\sigma\rho}\tilde{C}^{\alpha\beta}{}_{\sigma\rho})$. In view of (13) we have rewritten the initially obtained restrictions of the antisymmetric part of the Ricci tensor as the constraint on $D_{[\mu\nu]}$.

Hence the classification of the possible gravitational instantons (cf Christensen and Duff 1979) is more specialised in U_4.

(I) *General case*: all irreducible parts of torsion and curvature are non-trivial (admissible for $(0,0)$, $(\frac{1}{2},0)$, $(0,\frac{1}{2})$, $(\frac{1}{2},\frac{1}{2})$).

(IIa) *Self-dual torsion* and self-dual Ricci tensor: $Q_- = 0$, $\tilde{R}_- = 0$. There we have the 'fine structure':

(IIa.1)	$\tilde{C}_+ \neq 0, \tilde{C}_- \neq 0, \tilde{E}^{(S)} \neq 0, D^{(S)} \neq 0$	(only (0, 1)),
(IIa.2)	$\tilde{C}_+ \neq 0, \tilde{C}_- = 0, \tilde{E}^{(S)} \neq 0, D^{(S)} \neq 0$	(for $(0, B), B > 1$),
(IIa.3)	$\tilde{C}_+ \neq 0, \tilde{C}_- \neq 0, \tilde{E}^{(S)} + D^{(S)} = 0$	(for $(\frac{1}{2}, 1)$),
(IIa.4)	$\tilde{C}_+ = 0, \tilde{C}_- = 0, \tilde{E}^{(S)} + D^{(S)} = 0$	(for $(\frac{1}{2}, B), B > 1$).

(IIb) *Anti-self-dual torsion* and anti-self-dual Ricci tensor: $Q_+ = 0$, $\tilde{R}_+ = 0$. This case has analogous subclasses:

(IIb.1)	$\tilde{C}_+ \neq 0, \tilde{C}_- \neq 0, \tilde{E}^{(S)} \neq 0, D^{(S)} \neq 0$	(for (1, 0)),
(IIb.2)	$\tilde{C}_+ = 0, \tilde{C}_- \neq 0, \tilde{E}^{(S)} \neq 0, D^{(S)} \neq 0$	(for $(A, 0), A > 1$),
(IIb.3)	$\tilde{C}_+ \neq 0, \tilde{C}_- \neq 0, \tilde{E}^{(S)} - D^{(S)} = 0$	(for $(1, \frac{1}{2})$),
(IIb.4)	$\tilde{C}_+ = 0, \tilde{C}_- = 0, \tilde{E}^{(S)} - D^{(S)} = 0$	(for $(A, \frac{1}{2}), A > 1$).

(III) *Torsion-free spaces* $Q^{\lambda}{}_{.\mu\nu} = 0$ (and hence $D_{\mu\nu} = 0$). All these are Einstein spaces $E_{\mu\nu} = 0$ with various Weyl duality:

(III.1)	$C_+ \neq 0, C_- \neq 0$	(for (1, 1)),
(III.2a)	$C_+ \neq 0, C_- = 0$	(for $(1, B), B > 1$),
(III.2b)	$C_+ = 0, C_- \neq 0$	(for $(A, 1), A > 1$),
(III.3)	$C_+ = 0, C_- = 0$	(for $(A, B), A > 1, B > 1$).

Thus in order to generalise the index theorems one should first of all revise the Riemannian results for the non-trivial general case (I), while the other non-zero torsion cases are strongly restricted by self-duality conditions and demand a separate study. At the same time for $(A = 0, B = 0)$ and $(A \geqslant 1, B \geqslant 1)$ we are left with the well known Riemannian results.

4. Spin-$\frac{1}{2}$ example

Axial and conformal anomalies for the quantised Dirac fields in U_4 have already been obtained by Obukhov (1982, 1983). However, this was done in the four-spinor formalism. In our opinion it would also be useful to rederive our results within the framework of the above described arbitrary spin formalism in U_4, as this important example can help to clarify the general structure of the higher-spin self-adjoint second-order operators in U_4.

The field equations for the irreducible spinors $\varphi(\frac{1}{2}, 0)$ and $\chi(0, \frac{1}{2})$ according to (4) are

$$\mathrm{i}\overset{*}{\nabla}_{A.}{}^{B'}\varphi^{A} = m\chi^{B'}, \qquad \mathrm{i}\overset{*}{\nabla}{}^{A}{}_{.B'}\chi^{B'} = m\varphi^{A}.$$

Consequently the second-order equations (5)–(6) read now

$$(-\tilde{\nabla}_\mu\tilde{\nabla}^\mu + m^2 + \tfrac{1}{4}(\tilde{R} + D) + \tilde{\nabla}_\mu Q^\mu - Q_\mu Q^\mu)\varphi^A + (Q^{\alpha}{}_{.\mu\nu}\overset{(+)}{S}{}^{\mu\nu A}{}_{B} + 2Q^\alpha\delta^A{}_B)\tilde{\nabla}_\alpha\varphi^B$$
$$+ \tfrac{1}{2}(\tilde{R}_{\mu\nu} - \tilde{\nabla}_\mu Q_\nu + \tilde{\nabla}_\nu Q_\mu)\overset{(+)}{S}{}^{\mu\nu A}{}_{B}\varphi^B = 0,$$

$$(-\tilde\nabla_\mu\tilde\nabla^\mu+m^2+\tfrac14(\tilde R-D)+\tilde\nabla_\mu Q^\mu-Q_\mu Q^\mu)\chi^{A'}+(Q^\alpha{}_{.\mu\nu}\overset{(-)}{S}{}^{\mu\nu A'}{}_{B'}+2Q^\alpha\delta^{A'}{}_{B'})\tilde\nabla_\alpha\chi^{B'}$$
$$+\tfrac12(\tilde R_{\mu\nu}-\tilde\nabla_\mu Q_\nu+\tilde\nabla_\nu Q_\mu)\overset{(-)}{S}{}^{\mu\nu A'}{}_{B'}\chi^{B'}=0. \qquad (15)$$

As we see the spin-$\frac12$ operators have the form $\Delta=-\tilde\nabla_\mu\tilde\nabla^\mu+S^\mu\tilde\nabla_\mu+X$, where $S^\mu=-2Q^\mu{}_{.\alpha\beta}\Sigma^{\alpha\beta}+2Q^\mu$,

$$X=m^2-\tfrac12\Sigma^{\mu\nu}\tilde R_{ab\mu\nu}\Sigma^{ab}+(\tilde\nabla_\mu Q^\mu-Q_\mu Q^\mu+2\Sigma^{\mu\nu}\tilde\nabla_\mu Q_\nu).$$

The last terms in S^μ and X are absent in operators proposed by Goldthorpe (1980), Kimura (1981a, b), Barth and Christensen (1983), but it is these terms which provide the self-adjointness of the operators under consideration.

We would like to end this section by giving the explicit expression for the (Minakshisundaram–Seeley–De Witt) b_4 coefficients for the Dirac $(\frac12,0)$ and $(0,\frac12)$ fields in U_4, because these important quantities have never been given elsewhere. These coefficients (for details see e.g. Christensen and Duff 1979) determine the one-loop gravitational counterterms, the stress tensor trace anomaly and the axial current anomaly. Earlier (Obukhov 1983) we have obtained the general formula for $b_4(A,B)$ for the arbitrary spin field in U_4. With its help (equation (3.9) of Obukhov 1983) we get for the operator (15)

$$b_4(\tfrac14(1\pm1),\tfrac14(1\mp1))=(4\pi)^{-2}[\tfrac1{12}\nabla_\mu(N^\mu\mp K^\mu)+m^4+m^2(\tfrac16R-\tfrac12\check Q_\alpha\check Q^\alpha)$$
$$-\tfrac1{60}\nabla_\mu\nabla^\mu R+\tfrac1{144}R^2-\tfrac1{90}R_{\mu\nu}R^{\mu\nu}-\tfrac7{720}R_{\alpha\beta\mu\nu}R^{\alpha\beta\mu\nu}$$
$$-\tfrac1{48}F_{\mu\nu}F^{\mu\nu}\mp\tfrac1{96}\varepsilon^{\alpha\beta\mu\nu}(R_{\sigma\rho\alpha\beta}R^{\sigma\rho}{}_{..\mu\nu}+\tfrac12F_{\alpha\beta}F_{\mu\nu})], \qquad (16)$$

where we denoted $F_{\mu\nu}=\partial_\mu\check Q_\nu-\partial_\nu\check Q_\mu$, $K_\mu=(\nabla_\alpha\nabla^\alpha-6m^2+\frac14\check Q_\alpha\check Q^\alpha-\frac12R)\check Q_\mu$,

$$N^\mu=\tfrac12[\check Q^\mu\nabla_\alpha\check Q^\alpha-\check Q^\alpha\nabla_\alpha\check Q^\mu+\nabla_\mu(\check Q_\alpha\check Q^\alpha)],$$

and $\check Q_\mu=\varepsilon_{\mu\alpha\beta\gamma}Q^{\alpha\beta\gamma}$ is the axial trace of torsion.

The choice of the upper signs in (16) gives $b_4(\frac12,0)$ and of the lower ones $b_4(0,\frac12)$. As one can easily see, (16) agrees with the previously obtained anomalies of the massless Dirac fields derived in the four-spinor formalism.

5. Discussion and conclusion

Starting from the flat space action principle and the minimal coupling assumption, we have obtained the generalised (*self-adjoint*) field equations for an arbitrary spin in the Riemann–Cartan space–time. It turns out that the consistency restrictions imposed on the curvature and torsion are stronger than in the Riemannian case and the background torsion is almost ruled out.

After the completion of our calculations we became aware of the recent paper of Barth and Christensen (1983) in which they also attempted the construction of the general theory of arbitrary spin fields in U_4. Our results are different in several aspects. The crucial point of our method is the use of the action principle. Thus our equations are new. Moreover they are self-adjoint whereas the earlier proposed operators are not. This fact is very important because the correct form of the field operators and their self-adjointness play the central role in the calculation of quantum corrections for the quantised fields in curved space–times. Naturally the consistency conditions (14) also differ from that of Barth and Christensen (1983), though of course the general structure of the possible constraints is the same (note also that a different technique is used for their analysis).

In our opinion one can try to overcome the consistency problems in U_4 by assuming some kind of non-minimal coupling, analogous to the Riemannian examples (see attempts of Buchdahl (1958, 1962, 1982) in this direction). Another possible way out is to use supersymmetry. In fact the simple supergravity (see van Nieuwenhuizen 1981) can be considered as an example of consistent theory of the spin-$\frac{3}{2}$ field in U_4.

Appendix

Our conventions are as in (Obukhov 1983). Greek indices $\alpha, \beta, \mu, \nu, \ldots = 0, 1, 2, 3$ refer to the space–time coordinate basis $\{\partial_\mu\}$; Latin indices $a, b, c, \ldots = 0, 1, 2, 3$ refer to an arbitrary non-holonomic orthonormal basis in the tangent space; $A, B, \ldots = 1, 2$ are the two-spinor indices, while $A', B', \ldots$ refer to the complex conjugate two-spinors.

The space–time metric is $g_{\mu\nu} = h^a_\mu h^b_\nu \delta_{ab}$; the world affine connection $\tilde{\Gamma}^\alpha_{\beta\mu}$ defines the parallel transport $\delta A^\alpha = -\tilde{\Gamma}^\alpha_{\beta\mu} A^\beta \delta X^\mu$ and can be decomposed into the Riemannian connection (Christoffel symbols) $\Gamma^\alpha_{\beta\mu} = \frac{1}{2} g^{\alpha\nu}(\partial_\beta g_{\mu\nu} + \partial_\mu g_{\beta\nu} - \partial_\nu g_{\beta\mu})$ and contorsion as follows:

$$\tilde{\Gamma}^\alpha_{\beta\mu} = \Gamma^\alpha_{\beta\mu} + Q^\alpha_{\cdot\beta\mu} + Q_{\beta\mu\cdot}^{\ \ \alpha} + Q_{\mu\beta\cdot}^{\ \ \alpha}.$$

By the tilde we denote geometrical objects, constructed from the Riemann–Cartan connection $\tilde{\Gamma}^\alpha_{\beta\mu}$, while the Riemannian quantities have no additional marks.

The Riemann–Cartan curvature is

$$\tilde{R}^\alpha_{\cdot\beta\mu\nu} = \partial_\mu \tilde{\Gamma}^\alpha_{\beta\nu} - \partial_\nu \tilde{\Gamma}^\alpha_{\beta\mu} + \tilde{\Gamma}^\alpha_{\rho\mu} \tilde{\Gamma}^\rho_{\beta\nu} - \tilde{\Gamma}^\alpha_{\rho\nu} \tilde{\Gamma}^\rho_{\beta\mu};$$

its contractions are the Ricci tensor $\tilde{R}_{\mu\nu} = \tilde{R}^\alpha_{\cdot\mu\alpha\nu}$ and the curvature scalar $\tilde{R} = \tilde{R}_{\mu\nu} g^{\mu\nu}$.

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